

COVERING TECHNIQUES IN AUSLANDER-REITEN THEORY

CLAUDIA CHAIO, PATRICK LE MEUR, AND SONIA TREPODE

ABSTRACT. Given a finite dimensional algebra over a perfect field the text introduces covering functors over the mesh category of any modulated Auslander-Reiten component of the algebra. This is applied to study the composition of irreducible morphisms between indecomposable modules in relation with the powers of the radical of the module category.

INTRODUCTION

Let A be a finite-dimensional algebra over a field \mathbb{k} . The representation theory of A deals with the category $\text{mod } A$ of finitely generated (right) A -modules. In particular it aims at describing the indecomposable modules up to isomorphism and the morphisms between them. In this purpose the Auslander-Reiten theory gives useful tools such as irreducible morphisms and almost split sequences. These two particular concepts have been applied to study singularities of algebraic varieties and Cohen-Macaulay modules over commutative rings.

Let $\text{ind } A$ be the full subcategory of $\text{mod } A$ containing one representative of each isomorphism class of indecomposable A -modules. Given $X, Y \in \text{ind } A$, a morphism $f: X \rightarrow Y$ is called *irreducible* if it lies in $\text{rad} \setminus \text{rad}^2$. Here rad denotes the radical of the module category, that is, the ideal in $\text{mod } A$ generated by the non-isomorphisms between indecomposable modules. The powers rad^ℓ of the radical are recursively defined by $\text{rad}^{\ell+1} = \text{rad}^\ell \cdot \text{rad} = \text{rad} \cdot \text{rad}^\ell$. The Auslander-Reiten theory encodes part of the information of $\text{mod } A$ in the Auslander-Reiten quiver $\Gamma(\text{mod } A)$. This concentrates much of the combinatorial information on the irreducible morphisms and almost split sequences. However it does not give a complete information on the composition of two (or more) irreducible morphisms. For example the composition of n irreducible morphisms obviously lies in rad^n but it may lie in rad^{n+1} . It is proved in [IT84b, Thm. 13.3] that if these irreducible morphisms form a sectional path then their composition lies in $\text{rad}^n \setminus \text{rad}^{n+1}$. This result was made more precise for finite-dimensional algebras over algebraically closed fields in a study [CLMT11] of the degrees of irreducible morphisms (in the sense of [Liu92]) and their relationship to the representation type of the algebra. The results in [CLMT11] are based on well-behaved functors introduced first in [Rie80, BG82] for (self-injective) algebras of finite representation type. This text presents general constructions of well-behaved functors with application to composition of irreducible morphisms.

Date: December 2, 2014.

2010 Mathematics Subject Classification. 16G10, 16G60, 16G70.

The first and third named author thankfully acknowledges financial support from CONICET and Universidad Nacional de Mar del Plata, Argentina. The third named author is a CONICET researcher.

Let Γ be a connected component of the Auslander-Reiten quiver of A (or, an Auslander-Reiten component, for short). Let $\text{ind } \Gamma$ be the full subcategory of $\text{ind } A$ with set of objects the modules $X \in \text{ind } A$ lying in Γ . Beyond the combinatorial structure on Γ , the mesh-category $\mathbf{k}(\Gamma)$ is a first approximation of $\text{ind } \Gamma$ taking into account the composition of irreducible morphisms. Actually Igusa and Todorov have shown that Γ comes equipped with a \mathbf{k} -modulation ([IT84a]) which includes the division algebra $\kappa_X = \text{End}_A(X)/\text{rad}(X, X)$ and the $\kappa_X - \kappa_Y$ -bimodule $\text{irr}(X, Y) = \text{rad}(X, Y)/\text{rad}^2(X, Y)$ for every $X, Y \in \Gamma$. The category $\mathbf{k}(\Gamma)$ may be defined by generators and relations (see Section 1 for details). Its objects are the modules $X \in \Gamma$, the generators are the classes of morphisms $u \in \kappa_X$ (as morphisms in $\mathbf{k}(\Gamma)(X, X)$) and $u \in \text{irr}(X, Y)$ (as morphisms in $\mathbf{k}(\Gamma)(X, Y)$), for every $X, Y \in \Gamma$, and the ideal of relations is the mesh ideal.

When \mathbf{k} is a perfect field this text introduces a covering functor of $\text{ind } \Gamma$ in order to get information about the composition of irreducible morphisms in Γ .

The Auslander-Reiten component is called standard if there exists an isomorphism of categories $\mathbf{k}(\Gamma) \simeq \text{ind } \Gamma$. Not all Auslander-Reiten components are standard and in many cases there even exist no functor $\mathbf{k}(\Gamma) \rightarrow \text{ind } \Gamma$. For instance if Γ has oriented cycles then such a functor is likely not to exist. This may be bypassed replacing the mesh category $\mathbf{k}(\Gamma)$ by that of a suitable translation quiver $\tilde{\Gamma}$ with a \mathbf{k} -modulation such that there exists a covering $\pi: \tilde{\Gamma} \rightarrow \Gamma$. It appears that the composition of irreducible morphisms in $\text{ind } \Gamma$ may be studied using $\mathbf{k}(\tilde{\Gamma})$ provided that there exists a so-called well-behaved functor $\mathbf{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$. Let $(\kappa_x, M(x, y))_{x, y}$ be the \mathbf{k} -modulation of $\tilde{\Gamma}$. By definition $\kappa_x = \kappa_{\pi x}$ and $M(x, y) = \text{irr}(\pi x, \pi y)$ for every $x, y \in \tilde{\Gamma}$. Then a functor $F: \mathbf{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ is well-behaved if it induces isomorphisms $\kappa_x \simeq \kappa_{\pi x}$ and $M(x, y) \simeq \text{irr}(\pi x, \pi y)$, for every $x, y \in \tilde{\Gamma}$. The construction of F relies on three fundamental facts. Firstly, if one tries to construct such an F then it is quite natural to proceed by induction. The translation quiver $\tilde{\Gamma}$ is called *with length* if any two paths in $\tilde{\Gamma}$ having the same source and the same target have the same length. As mentioned above an inductive construction is likely not to work if $\tilde{\Gamma}$ has oriented cycles and actually simple examples show that this construction fails if $\tilde{\Gamma}$ is not with length. Note that $\tilde{\Gamma}$ is with length when $\tilde{\Gamma}$ is the universal cover of [BG82]. Secondly, if $x \in \tilde{\Gamma}$ then the ring homomorphism $\kappa_x \hookrightarrow \mathbf{k}(\tilde{\Gamma})(x, x) \xrightarrow{F} \text{End}_A(\pi x)$ is a section of the quotient homomorphism $\text{End}_A(X) \twoheadrightarrow \kappa_x$. In view of the Wedderburn-Malcev theorem this section is most likely to exist in the framework of algebras over perfect fields. Finally, given an irreducible morphism $f: X \rightarrow Y$ with $X, Y \in \Gamma$ then there exist $x, y \in \tilde{\Gamma}$ and $u \in \mathbf{k}(\tilde{\Gamma})(x, y)$ such that $f - Fu \in \text{rad}^2$. In view of studying the composition of irreducible morphisms in $\text{ind } \Gamma$ one may wish to have an equality $f = Fu$. This would permit to *lift* the study into $\mathbf{k}(\tilde{\Gamma})$ where the composition of morphisms is better understood because of the mesh ideal. Keeping in mind these comments the main result of this text is the following.

Theorem A. *Let A be a finite-dimensional algebra over a perfect field \mathbf{k} . Let Γ be an Auslander-Reiten component of A . Let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers where $\tilde{\Gamma}$ is with length. There exists a well-behaved functor $F: \mathbf{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$.*

The study of the composition of irreducible morphisms in $\text{ind } \Gamma$ using such a covering functor F is made possible by the following lifting (or, covering) property of F which is the second main result of the text. No assumption is made on length.

Theorem B. *Let A be a finite-dimensional algebra over a perfect field \mathbb{k} . Let Γ be an Auslander-Reiten component of A . Let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers. Let $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ be a well-behaved functor, $x, y \in \tilde{\Gamma}$ and let $n \geq 0$.*

(a) *The two following maps induced by F are bijective*

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})(x, z) / \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})(x, z) \rightarrow \text{rad}^n(Fx, Fy) / \text{rad}^{n+1}(Fx, Fy)$$

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})(z, x) / \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})(z, x) \rightarrow \text{rad}^n(Fy, Fx) / \text{rad}^{n+1}(Fy, Fx) \quad .$$

(b) *The two following maps induced by F are injective*

$$\bigoplus_{Fz=Fy} \mathbb{k}(\tilde{\Gamma})(x, z) \rightarrow \text{Hom}_A(Fx, Fy) \quad \text{and} \quad \bigoplus_{Fz=Fy} \mathbb{k}(\tilde{\Gamma})(z, x) \rightarrow \text{Hom}_A(Fy, Fx).$$

(c) *Γ is generalized standard if and only if F is a covering functor, that is, the two maps of (b) are bijective (see [BG82, 3.1]).*

Here $\mathfrak{R} \mathbb{k}(\tilde{\Gamma})$ is the ideal in $\mathbb{k}(\tilde{\Gamma})$ generated by the morphisms in $M(x, y)$, for $x, y \in \tilde{\Gamma}$. Call it the *radical* of $\mathbb{k}(\tilde{\Gamma})$ by abuse of terminology. Define its powers $\mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})$ like for the radical of $\text{mod } A$. Here is an interpretation of Theorem B. Both $\mathbb{k}(\tilde{\Gamma})$ and $\text{ind } \Gamma$ are filtered by the powers of their respective radicals. The above theorem asserts that F induces a covering functor $\text{gr } \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{gr } \text{ind } \Gamma$ (in the sense of [BG82]) between the associated graded categories.

This text is therefore organised as follows. Section 1 is a reminder on basic results on irreducible morphisms, modulated translation quivers and their mesh-categories, and coverings of translation quivers. Section 2 proves the above theorems. Section 3 gives an application to the composition of irreducible morphisms.

In the sequel \mathbb{k} denotes a perfect field. Hence the tensor product over \mathbb{k} of two finite-dimensional division algebras is semi-simple. Also if R is a finite-dimensional \mathbb{k} -algebra and $J \subseteq R$ is a two-sided ideal such that R/J is a division \mathbb{k} -algebra, then the natural surjection $R \twoheadrightarrow R/J$ admits a section $R/J \hookrightarrow R$ as a \mathbb{k} -algebra.

1. PRELIMINARIES

1.1. Notation on modules. Let A be a finite dimensional \mathbb{k} -algebra. Given modules $X, Y \in \text{ind } A$, the quotient vector space $\text{rad}(X, Y) / \text{rad}^2(X, Y)$ is denoted by $\text{irr}(X, Y)$ and called the *space of irreducible morphisms* from X to Y . It is naturally an $\text{End}_A(X) / \text{rad}(X, X) - \text{End}_A(Y) / \text{rad}(Y, Y)$ -bimodule. The division \mathbb{k} -algebra $\text{End}_A(X) / \text{rad}(X, X)$ is denoted by κ_X . Let $X, Y \in \text{ind } A$ be distinct. If $u \in \text{End}_A(X)$ (or $v \in \text{rad}(X, Y)$) then $\bar{u} \in \kappa_X$ (or $\bar{v} \in \text{irr}(X, Y)$) denotes the residue class of u modulo rad (or of v modulo rad^2 , respectively). Recall that the *Auslander-Reiten quiver* of A is the quiver $\Gamma(\text{mod } A)$ with vertices the modules in $\text{ind } A$, such that there is an arrow (and exactly one) $X \rightarrow Y$ if and only if $\text{irr}(X, Y) \neq 0$ for every pair of vertices $X, Y \in \Gamma$. Let $f: X \rightarrow \bigoplus_{i=1}^r X_i^{n_i}$ be an irreducible morphism where $X, X_1, \dots, X_r \in \text{ind } A$ are pairwise non isomorphic and $n_1, \dots, n_r \geq 1$. Then f is called *strongly irreducible* if, for every $i \in \{1, \dots, r\}$, the

n_i -tuple $(\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}})$ of $\text{irr}(X, X_i)$ is free over $\kappa_X \otimes_{\mathbb{k}} \kappa_{X_i}^{\text{op}}$. This tuple is always free over $\kappa_{X_i}^{\text{op}}$. Also, f is strongly irreducible under any of the following conditions: if $n_1 = \dots = n_r = 1$ (then $\overline{f_{i,1}} \in \text{irr}(X, X_i)$ is non zero, and hence free over $\kappa_X \otimes_{\mathbb{k}} \kappa_{X_i}$); if $\kappa_X \simeq \mathbb{k}$; or if \mathbb{k} is algebraically closed (then $\kappa_X \simeq \mathbb{k}$). Finally, if f has finite left degree (in the sense of [Liu92]) then $d_l(X \rightarrow X_i) < \infty$ and the arrow $X \rightarrow X_i$ in Γ has valuation $(1, \alpha')$ or $(\alpha, 1)$ ([Liu92, 1.7]). In this case $\text{irr}(X, X_i)$ has rank 1 as a $\kappa_X - \kappa_{X_i}$ -bimodule. Thus, in such a case, being strongly irreducible for f is equivalent to satisfying $n_1 = \dots = n_r = 1$.

1.2. Factorisation through minimal almost split morphisms. The reader is referred to [ARO97] for basics on Auslander-Reiten theory. In the sequel the factorisation property of minimal almost split morphisms is used as follows.

Lemma. *Let $u: X \rightarrow Y$ be a left minimal almost split morphism. Let $Z \in \text{mod } A$. Let $v \in \text{rad}^{n+1}(X, Z)$ for some $n \geq 0$. There exists $w \in \text{rad}^n(Y, Z)$ such that $v = uw$.*

Proof. By definition there is a decomposition $v = \sum_i v'_i v''_i$ where the v'_i lie in rad , the v''_i lie in rad^n and i runs through some index set. Then the conclusion follows from the fact that each v'_i factors through u . \square

1.3. Modulated translation quivers and their mesh-categories. Let Γ be a *translation quiver*, that is, Γ is a quiver with no loops and no multiple arrows endowed with two distinguished set of vertices the elements of which are called *projectives* and *injectives*, respectively, and endowed with a bijection $x \mapsto \tau x$ (called the *translation*) from the set of non-projective vertices to the set of non-injective vertices, such that for every vertices x, y with x non-projective, there is an arrow $y \rightarrow x$ if and only if there is an arrow $\tau x \rightarrow y$. All translation quivers are assumed to be *locally finite*: Every vertex is connected to at most finitely many arrows. Auslander-Reiten quivers and Auslander-Reiten components are translation quivers (with translation equal to the Auslander-Reiten translation). Given a non-projective vertex x , the subquiver of Γ formed by the arrows starting in τx and the arrows arriving in x is called the *mesh* starting in τx .

A (\mathbb{k}) -*modulation* on Γ is the following data

- (i) a division \mathbb{k} -algebra κ_x for every vertex $x \in \Gamma$,
- (ii) a non-zero $\kappa_x - \kappa_y$ bimodule $M(x, y)$ for every arrow $x \rightarrow y$ in Γ ,
- (iii) a \mathbb{k} -algebra isomorphism $\tau_*: \kappa_x \xrightarrow{\sim} \kappa_{\tau x}$ for every vertex $x \in \Gamma$,
- (iv) a non-degenerate $\kappa_y - \kappa_x$ -linear map $\sigma_*: M(y, x) \otimes_{\kappa_x} M(\tau x, y) \rightarrow \kappa_y$ (the left κ_x -module structure on $M(\tau x, y)$ is defined using its structure of left $\kappa_{\tau x}$ -module and $\tau_*: \kappa_x \rightarrow \kappa_{\tau x}$).

With such a structure, Γ is called a *modulated translation quiver*. If A is a finite-dimensional algebra over a field \mathbb{k} then the Auslander-Reiten quiver $\Gamma(\text{mod } A)$ has a \mathbb{k} -modulation as follows ([IT84a, 2.4, 2.5]). For every non-projective $X \in \text{ind } A$ fix an almost split sequence $0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod } A$. Then

- $\kappa_X = \text{End}_A(X)/\text{rad}(X, X)$ for every $X \in \text{ind } A$,
- $M(X, Y) = \text{irr}(X, Y)$ for every arrow $X \rightarrow Y$ in $\Gamma(\text{mod } A)$,
- for every $X \in \text{ind } A$ and every morphism $u: X \rightarrow X$ defining the residue class $\overline{u} \in \kappa_X$, let $\tau_* \overline{u}: \tau_A X \rightarrow \tau_A X$ be the residue class \overline{v} where $v: \tau_A X \rightarrow$

$\tau_A X$ is a morphism fitting into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow v & & \downarrow & & \downarrow u \\ 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X \longrightarrow 0, \end{array}$$

- let $X, Y \in \text{ind } A$ with X non-projective and assume that there is an arrow $Y \rightarrow X$ in $\Gamma(\text{mod } A)$. Let $\bar{u} \in M(\tau_A X, Y)$ and $\bar{v} \in M(Y, X)$ be residue classes of morphisms $u: \tau_A X \rightarrow Y$ and $v: Y \rightarrow X$, respectively. Then define $\sigma_*(\bar{v} \otimes \bar{u})$ as the composition $\overline{v'u'}$ where u', v' are morphisms fitting into a commutative diagram

$$\begin{array}{ccccccc} & & & & Y & & \\ & & & & \downarrow v & & \\ & & & & X & \longrightarrow & 0 \\ & & & \swarrow v' & \uparrow & & \\ 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow u & & \swarrow u' & & \\ & & Y & & & & \end{array}$$

This construction does not depend on the initial choice of the almost split sequences up to an isomorphism of modulated translation quivers ([IT84a, 2.5]). In the sequel $\Gamma(\text{mod } A)$ is considered as a modulated translation quiver as above.

If Γ is a modulated translation quiver, its *mesh-category* $\mathbb{k}(\Gamma)$ is defined as follows ([IT84a, 1.7]). Let S be the semi-simple category whose object set is the set of vertices in Γ and such that $S(x, y) = \kappa_x$ if $x = y$ and $S(x, y) = 0$ otherwise. The collection $\{M(x, y)\}_{x \rightarrow y \text{ in } \Gamma}$ naturally defines an $S - S$ -bimodule denoted by M . The *path-category* is the tensor category $T_S(M)$ also denoted by $\mathbb{k}\Gamma$. The *mesh-ideal* is the ideal in $\mathbb{k}\Gamma$ generated by a collection of morphisms $\gamma_x: \tau x \rightarrow x$ indexed by the non-projective vertices $x \in \Gamma$. Given a non-projective vertex $x \in \Gamma$, a morphism $\gamma_x: \tau x \rightarrow x$ in $\mathbb{k}\Gamma$ is defined as follows. For every arrow $y \rightarrow x$ ending in x , fix a basis (u_1, \dots, u_d) of the κ_y -vector space $M(y, x)$. Let (u_1^*, \dots, u_d^*) be the associated dual basis of the κ_y -vector space $M(\tau x, y)$ under the pairing σ_* (that is, $\sigma_*(u_i \otimes u_j^*)$ is 1 if $i = j$ and 0 otherwise). Then $\gamma_x = \sum_{y \rightarrow x \text{ in } \Gamma} \sum_i u_i^* u_i \in \mathbb{k}\Gamma(\tau x, x)$. This morphism

does not depend on the choice of the basis (u_1, \dots, u_d) . The mesh-category is then defined as the quotient category of $\mathbb{k}\Gamma$ by the mesh-ideal.

Let Γ be a component of $\Gamma(\text{mod } A)$ endowed with a modulation as above. As proved in [IT84a, Sect. 2], the mesh-category $\mathbb{k}(\Gamma)$ does not depend on the choice of the almost split sequences used to define the modulation up to an isomorphism of \mathbb{k} -linear categories. The following lemma explains how to recover the mesh-relations γ_X and the pairing σ_* starting from a different choice of almost split sequences.

Lemma. *In the previous setting, let $X \in \Gamma$ be non-projective and let*

$$0 \rightarrow \tau_A X \xrightarrow{\alpha} \bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{\beta} X \rightarrow 0$$

be the almost split sequence ending in X that is used in the definition of the modulation on Γ , where $X_1, \dots, X_r \in \Gamma$ are pairwise distinct. Let

$$0 \rightarrow \tau_A X \xrightarrow{f = \begin{bmatrix} f_{i,j} ; & 1 \leq i \leq r \\ & 1 \leq j \leq n_i \end{bmatrix}^t} \bigoplus_i X_i^{n_i} \xrightarrow{g = \begin{bmatrix} g_{i,j} ; & 1 \leq i \leq r \\ & 1 \leq j \leq n_i \end{bmatrix}} X \rightarrow 0$$

be another almost split sequence where the $f_{i,j}: \tau_A X \rightarrow X_i$ and the $g_{i,j}: X_i \rightarrow X$ are the components of f and g , respectively. Then there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau_A X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & X \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \parallel \\ 0 & \longrightarrow & \tau_A X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & X \longrightarrow 0 \end{array}$$

where u, v are isomorphisms. With this data:

- (a) $(\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}})$ is a basis of the left κ_{X_i} -module $\text{irr}(X_i, X)$ and the corresponding dual basis of the right κ_{X_i} -module $\text{irr}(\tau_A X, X_i)$ under the pairing σ_* is $(\overline{uf_{i,1}}, \dots, \overline{uf_{i,n_i}})$, for every $i \in \{1, \dots, r\}$,
- (b) $\gamma_X = \sum_{i,j} \overline{uf_{i,j}} \otimes \overline{g_{i,j}}$.

Proof. (b) follows directly from (a) and from the definition of γ_X . It therefore suffices to prove (a). The existence of f and g are direct consequences of the basic properties of almost split sequences. Also, for every i , the given n_i -tuples are indeed bases because they arise from a left (or right) minimal almost split morphism. Let $w: \bigoplus_i X_i^{n_i} \rightarrow \bigoplus_i X_i^{n_i}$ be v^{-1} . For every $i, i' \in \{1, \dots, r\}$, and every $1 \leq j \leq n_i$, and every $1 \leq j' \leq n_{i'}$, let $v_{(i,j),(i',j')}: X_i \rightarrow X_{i'}$ and $w_{(i,j),(i',j')}: X_i \rightarrow X_{i'}$ be the respective components of v and w , from the j -th component X_i of $X_i^{n_i}$ to the j' -th component $X_{i'}$ of $X_{i'}^{n_{i'}}$. Then, the equality $wv = \text{Id}$ reads

$$(1) \quad \forall (i,j), (i'',j'') \quad \sum_{(i',j')} w_{(i,j),(i',j')} v_{(i',j'),(i'',j'')} = \begin{cases} \text{Id}_{X_i} & \text{if } (i,j) = (i'',j'') \\ 0 & \text{otherwise.} \end{cases}$$

The morphisms u, v, w therefore yield commutative diagrams for every $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\}$:

$$\begin{array}{ccc} \tau_A X & \xrightarrow{\alpha} & \bigoplus_{i'} X_{i'}^{n_{i'}} \\ \downarrow uf_{i,j} & \swarrow [v_{(i',j'),(i,j)}; (i',j')]^t & \\ X_i & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & X_i \\ & \swarrow [w_{(i,j),(i',j')}; (i',j')] & \downarrow g_{i,j} \\ \bigoplus_{i'} X_{i'}^{n_{i'}} & \xrightarrow{\beta} & X \end{array}.$$

Thus $\sigma_* (\overline{g_{i,j}} \otimes \overline{uf_{i'',j''}}) = \sum_{i',j'} w_{(i,j),(i',j')} v_{(i',j'),(i'',j'')} \text{ for every } (i,j) \text{ and } (i'',j'').$

This and (1) show (a). \square

1.4. Radical in mesh-categories. Let $\tilde{\Gamma}$ be a modulated translation quiver. Recall that the radical $\mathfrak{Rk}(\tilde{\Gamma})$ of $\mathbb{k}(\tilde{\Gamma})$ was defined in the introduction. For every arrow $x \rightarrow y$ in $\tilde{\Gamma}$ the natural map $M(x,y) \rightarrow \mathbb{k}(\tilde{\Gamma})(x,y)$ is one-to-one. And the $\kappa_x - \kappa_y$ -bimodule $\mathfrak{Rk}(\tilde{\Gamma})(x,y)$ decomposes as $M(x,y) \oplus \mathfrak{R}^2 \mathbb{k}(\tilde{\Gamma})(x,y)$. The description of the ideal $\mathfrak{R}^\ell \mathbb{k}(\tilde{\Gamma})$ is easier when $\tilde{\Gamma}$ is with length as shows the following proposition. It is central in this text and used without further reference. The proof is a small variation of [Cha10, 2.1] where the description was first proved in the case $\kappa_x = \mathbb{k}$ for every vertex $x \in \tilde{\Gamma}$.

Proposition. *Let $\tilde{\Gamma}$ be a translation quiver with length and $x, y \in \tilde{\Gamma}$. If there is a path of length ℓ from x to y in $\tilde{\Gamma}$, then:*

- (a) $\mathbb{k}(\tilde{\Gamma})(x,y) = \mathfrak{Rk}(\tilde{\Gamma})(x,y) = \mathfrak{R}^2 \mathbb{k}(\tilde{\Gamma})(x,y) = \dots = \mathfrak{R}^\ell \mathbb{k}(\tilde{\Gamma})(x,y)$.

(b) $\Re^i \mathbb{k}(\tilde{\Gamma})(x, y) = 0$ if $i > \ell$.

1.5. Coverings of translation quivers ([BG82, 1.3]). A *covering of translation quivers* is a quiver morphism $p: \tilde{\Gamma} \rightarrow \Gamma$ such that

- (a) $\Gamma, \tilde{\Gamma}$ are translation quivers, Γ is connected,
- (b) a vertex $x \in \tilde{\Gamma}$ is projective (or injective, respectively) if and only if so is px ,
- (c) p commutes with the translations in Γ and $\tilde{\Gamma}$ (where these are defined),
- (d) for every vertex $x \in \tilde{\Gamma}$ the map $\alpha \mapsto p(\alpha)$ induces a bijection from the set of arrows in $\tilde{\Gamma}$ starting in x (or ending in x) to the set of arrows in Γ starting in px (or ending in px , respectively).

Let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers. If Γ is modulated by division \mathbb{k} -algebras κ_x and bimodules $M(x, y)$ for every vertex x and every arrow $x \rightarrow y$, then $\tilde{\Gamma}$ is modulated by the division \mathbb{k} -algebra $\kappa_x := \kappa_{\pi x}$ at the vertex $x \in \tilde{\Gamma}$ and by the bimodule $M(\pi x, \pi y)$ for every arrow $x \rightarrow y$ in $\tilde{\Gamma}$. When $\tilde{\Gamma}$ is with length, it has *length function*, that is, a map $x \mapsto \ell(x)$ defined on vertices such that $\ell(y) = \ell(x) + 1$ for every arrow $x \rightarrow y$. See [BG82, 1.6] for the construction of such a function in the particular case where $\tilde{\Gamma}$ is the universal cover of Γ ([BG82, 1.2, 1.3]). Note that quivers have no parallel arrows here hence the notion of universal cover coincides with that of *generic cover* used in [CLMT11]).

From now on, A is a finite-dimensional \mathbb{k} -algebra. Its Auslander-Reiten components and their coverings are modulated as in 1.3 and above, respectively. To avoid possible confusions, upper-case letters (X, Y, \dots) stand for vertices in Auslander-Reiten components and lower-case letters (x, y, \dots) stand for vertices in other translation quivers. But the same notation (κ, M) is used for all \mathbb{k} -modulations.

2. WELL-BEHAVED FUNCTORS

Let Γ be an Auslander-Reiten component of A and $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers. This section introduces the notion of well-behaved functors $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ and proves their existence when $\tilde{\Gamma}$ is with length. Such objects were first introduced over algebraically closed fields for Auslander-Reiten quivers of algebras of finite representation type ([Rie80, Sect. 1] and [BG82, Sect. 2]). The section also proves the lifting properties of well-behaved functors (see [CLMT11, Sect. 2] when \mathbb{k} is algebraically closed).

Until the end of 2.5 the following convention is implicitly used. If $f: X \rightarrow \bigoplus_{i=1}^r X_i^{n_i}$ is an irreducible morphism it is assumed that $X \in \Gamma$, that $X_1, \dots, X_r \in \Gamma$ are pairwise distinct, and $n_1, \dots, n_r \geq 1$. Then f is written $f = [f_{i,j}; i, j]$.

2.1. Sections of residue fields and of spaces of irreducible morphisms. Let $X \in \Gamma$ be a vertex. For a given section $\kappa_X = \text{End}_A(X)/\text{rad}(X, X) \rightarrow \text{End}_A(X)$ of the \mathbb{k} -algebra surjection $\text{End}_A(X) \rightarrow \text{End}_A(X)/\text{rad}(X, X)$, the image is denoted by \mathbb{k}_X . Then $\mathbb{k}_X \subseteq \text{End}_A(X)$ is a subalgebra such that $\text{End}_A(X) = \mathbb{k}_X \oplus \text{rad}(X, X)$ as a \mathbb{k} -vector space and the surjection $\text{End}_A(X) \twoheadrightarrow \kappa_X$ restricts to a \mathbb{k} -algebra isomorphism $\mathbb{k}_X \xrightarrow{\sim} \kappa_X$. For short, \mathbb{k}_X is called a *section of κ_X* . There always exists such a (non unique) \mathbb{k}_X by the Wedderburn-Malcev Theorem.

Let $X \rightarrow Y$ be an arrow in Γ . Then $X \not\cong Y$ and $\text{Hom}_A(X, Y) = \text{rad}(X, Y)$. Suppose given sections $\mathbb{k}_X \subseteq \text{End}_A(X)$ and $\mathbb{k}_Y \subseteq \text{End}_A(Y)$ of κ_X and κ_Y , respectively. Then $\text{irr}(X, Y)$ is a $\mathbb{k}_X - \mathbb{k}_Y$ -bimodule using the isomorphisms $\mathbb{k}_X \xrightarrow{\sim} \kappa_X$

and $\mathbb{k}_Y \xrightarrow{\sim} \kappa_Y$. By a $\mathbb{k}_X - \mathbb{k}_Y$ -linear section (of $\text{irr}(X, Y)$) is meant a section $\text{irr}(X, Y) \rightarrow \text{rad}(X, Y)$ of the canonical surjection $\text{rad}(X, Y) \twoheadrightarrow \text{irr}(X, Y)$ in the category of $\mathbb{k}_X - \mathbb{k}_Y$ -bimodules. Such a section always exists because the \mathbb{k} -algebra $\mathbb{k}_X \otimes_{\mathbb{k}} \mathbb{k}_Y^{op}$ is semisimple. Note that the datum of a linear section depends on the choice of the algebra sections \mathbb{k}_X and \mathbb{k}_Y .

2.2. Well-behaved functors. The following definition extends to the case of perfect fields the already known definition of well-behaved functors when the base field is algebraically closed ([BG82, 3.1] and [Rie80, 2.2]). A \mathbb{k} -linear functor $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ is called *well-behaved* if

- (a) $Fx = \pi x$ for every vertex $x \in \tilde{\Gamma}$,
- (b) for every vertex $x \in \tilde{\Gamma}$, the \mathbb{k} -algebra map from $\kappa_{\pi x} = \kappa_x$ to $\text{End}_A(\pi x)$ defined by $u \mapsto F(u)$ is a section of the natural surjection $\text{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x}$. Its image is denoted by \mathbb{k}_x ,
- (c) for every arrow $x \rightarrow y$ in $\tilde{\Gamma}$, the following \mathbb{k} -linear composite map is a $\mathbb{k}_x - \mathbb{k}_y$ -linear section in the obvious way

$$\text{irr}(\pi x, \pi y) = M(x, y) \hookrightarrow \mathbb{k}(\tilde{\Gamma})(x, y) \xrightarrow{F} \text{Hom}_A(\pi x, \pi y).$$

Note that if F is as in the definition then distinct vertices $x, x' \in \tilde{\Gamma}$ such that $\pi x = \pi x'$ may give rise to different sections \mathbb{k}_x and $\mathbb{k}_{x'}$ in $\text{End}_A(\pi x)$. The data of a section $\mathbb{k}_x \subseteq \text{End}_A(\pi x)$ of $\kappa_{\pi x}$, for every vertex $x \in \tilde{\Gamma}$, and that of a $\mathbb{k}_x - \mathbb{k}_y$ -linear section $M(x, y) \rightarrow \text{Hom}_A(\pi x, \pi y)$, for every arrow $x \rightarrow y$ in $\tilde{\Gamma}$ determine a unique \mathbb{k} -linear functor $\mathbb{k}\tilde{\Gamma} \rightarrow \text{ind } \Gamma$. It induces a \mathbb{k} -linear functor $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ if and only if it vanishes on $\gamma_x = \gamma_{\pi x}$ for every non-projective vertex $x \in \tilde{\Gamma}$. In such a case, F is well-behaved. Moreover, any well-behaved functor arises in this way.

2.3. Local sections on almost split sequences. The existence of well-behaved functors is based on the following technical lemma. It aims at constructing sections that are compatible with the modulation on Γ , in some sense.

Lemma. *Let $X \in \Gamma$ be a non-injective vertex and let*

$$0 \rightarrow X \xrightarrow{f} \bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{g} \tau_A^{-1} X \rightarrow 0$$

be an almost split sequence. Let $\mathbb{k}_X \subseteq \text{End}_A(X)$ and $\mathbb{k}_{X_i} \subseteq \text{End}_A(X_i)$ (for every $i \in \{1, \dots, r\}$) be sections of κ_X and κ_{X_i} respectively, and let $\text{irr}(X, X_i) \hookrightarrow \text{rad}(X, X_i)$ be a $\mathbb{k}_X - \mathbb{k}_{X_i}$ -linear section which maps $\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}}$ to $f_{i,1}, \dots, f_{i,n_i}$ respectively, for every $i \in \{1, \dots, r\}$. There exists a section $\mathbb{k}_{\tau_A^{-1}X} \hookrightarrow \text{End}_A(\tau_A^{-1}X)$ of $\kappa_{\tau_A^{-1}X}$ and a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1}X}$ -linear section $\text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{rad}(X_i, \tau_A^{-1}X)$ (for every $i \in \{1, \dots, r\}$) such that

- (a) *it maps $\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}}$ to $g_{i,1}, \dots, g_{i,n_i}$ respectively, for every $i \in \{1, \dots, r\}$,*
- (b) *the induced map $\bigoplus_{i=1}^r \text{irr}(X, X_i) \otimes_{\kappa_{X_i}} \text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{Hom}_A(X, \tau_A^{-1}X)$ vanishes on $\gamma_{\tau_A^{-1}X}$.*

Proof. Note that if such sections do exist then $(g_{i,1}, \dots, g_{i,n_i})$ must be a basis over \mathbb{k}_{X_i} of the image of $\text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{rad}(X_i, \tau_A^{-1}X)$ because g is a right minimal almost split morphism. In particular, the section $\mathbb{k}_{\tau_A^{-1}X} \subseteq \text{End}_A(\tau_A^{-1}X)$ must be

such that the left \mathbb{k}_{X_i} -submodule of $\text{rad}(X_i, \tau_A^{-1}X)$ generated by $g_{i,1}, \dots, g_{i,n_i}$ is a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1}X}$ -submodule of $\text{rad}(X_i, \tau_A^{-1}X)$, for every $i \in \{1, \dots, r\}$. The proof therefore proceeds as follows: 1) define a section $\mathbb{k}_{\tau_A^{-1}X} \subseteq \text{End}_A(\tau_A^{-1}X)$ satisfying this last condition, 2) define sections $\text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{rad}(X_i, \tau_A^{-1}X)$ so that (a) is satisfied, and 3) prove (b).

1) Let $\varphi \in \text{End}_A(\tau_A^{-1}X)$ and define a new representative $\varphi_1 \in \text{End}_A(\tau_A^{-1}X)$ of $\overline{\varphi} \in \kappa_{\tau_A^{-1}X}$ as follows. Let $0 \rightarrow X \xrightarrow{\alpha} \bigoplus_i X_i^{n_i} \xrightarrow{\beta} \tau_A^{-1}X \rightarrow 0$ be the almost split sequence used in the definition of the modulation on Γ . So there exists an isomorphism of exact sequences

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X \longrightarrow 0 \end{array} .$$

There also exists a commutative diagram

$$(3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow \theta & & \downarrow \varphi \\ 0 & \longrightarrow & X & \xrightarrow{\tilde{f}} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X \longrightarrow 0 \end{array}$$

for some morphisms θ and ψ . This yields the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X \longrightarrow 0 \\ & & \downarrow u\psi u^{-1} & & \downarrow v\theta v^{-1} & & \downarrow \varphi \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X \longrightarrow 0 \end{array} .$$

Therefore, $\tau_*(\overline{\varphi}) = \overline{u\psi u^{-1}} \in \kappa_X$. Now let $\psi_1 \in \mathbb{k}_X$ be the representative of ψ , that is, $\overline{\psi_1} = \overline{\psi}$. Since the section $\text{irr}(X, X_i) \rightarrow \text{rad}(X, X_i)$ is $\mathbb{k}_X - \mathbb{k}_{X_i}$ -linear and maps $\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}}$ to $f_{i,1}, \dots, f_{i,n_i}$, respectively, and since $(\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}})$ is a basis of the right \mathbb{k}_{X_i} -module $\text{irr}(X, X_i)$, there is a unique matrix $\eta_i \in M_{n_i}(\mathbb{k}_{X_i})$, considered as an endomorphism of $X_i^{n_i}$, making the following square commute for $i \in \{1, \dots, r\}$

$$\begin{array}{ccc} X & \xrightarrow{[f_{i,1}, \dots, f_{i,n_i}]} & X_i^{n_i} \\ \psi_1 \downarrow & & \downarrow \eta_i \\ X & \xrightarrow{[f_{i,1}, \dots, f_{i,n_i}]} & X_i^{n_i} \end{array} .$$

Therefore there exists a unique $\varphi_1 \in \text{End}_A(\tau_A^{-1}X)$ making the following diagram commute where $\eta: \bigoplus_i X_i^{n_i} \rightarrow \bigoplus_i X_i^{n_i}$ is the morphism defined by $\{\eta_i: X_i^{n_i} \rightarrow X_i^{n_i}\}_i$.

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X \longrightarrow 0 \\ & & \downarrow \psi_1 & & \downarrow \eta & & \downarrow \varphi_1 \\ 0 & \longrightarrow & X & \xrightarrow{\tilde{f}} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X \longrightarrow 0 \end{array} .$$

Hence the following diagram commutes

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1} X \longrightarrow 0 \\
& & \downarrow u\psi_1 u^{-1} & & \downarrow v\eta v^{-1} & & \downarrow \varphi_1 \\
0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1} X \longrightarrow 0
\end{array}$$

This entails $\overline{\varphi_1} = \tau_*^{-1}(\overline{u\psi_1 u^{-1}})$. But $\overline{\psi_1} = \overline{\psi}$ so that $\overline{u\psi_1 u^{-1}} = \overline{u\psi u^{-1}}$. Thus $\overline{\varphi_1} = \tau_*^{-1}(\overline{u\psi u^{-1}}) = \overline{\varphi}$. This construction therefore yields a well-defined map

$$\begin{aligned}
s: \quad \text{End}_A(\tau_A^{-1} X) &\rightarrow \text{End}_A(\tau_A^{-1} X) \\
\varphi &\mapsto \varphi_1 \quad (\text{such that } \overline{\varphi_1} = \overline{\varphi}).
\end{aligned}$$

Since $\tau_*: \kappa_{\tau_A^{-1} X} \rightarrow \kappa_X$ is a \mathbb{k} -algebra isomorphism and η_1, \dots, η_r are determined by ψ_1 (and the fixed \mathbb{k} -algebra sections), the map s is a \mathbb{k} -algebra homomorphism. Moreover if $\varphi \in \text{rad}(\tau_A^{-1} X, \tau_A^{-1} X)$ then $\overline{\varphi} = 0$ and the representative ψ_1 in \mathbb{k}_X of $\tau_*(\overline{\varphi}) = 0$ is 0; therefore, $\eta_1, \dots, \eta_r = 0$ and $\varphi_1 = 0$; in other words s vanishes on $\text{rad}(\tau_A^{-1} X, \tau_A^{-1} X)$. Finally if $\varphi' \in \text{End}_A(\tau_A^{-1} X)$ lies in the image of s then $\varphi' = s(\varphi) = \varphi_1$ for some $\varphi \in \text{End}_A(\tau_A^{-1} X)$; then, keeping the same notation for the morphisms $(\theta, \psi, \eta_1, \dots, \eta_r, \eta)$ used to define $s(\varphi)$ and using dashed notation for the corresponding morphisms used to define $s(\varphi')$, it follows that $\theta' = \eta$ and $\psi' = \psi_1 \in \mathbb{k}_X$; hence the representative $\psi'_1 \in \mathbb{k}_X$ of $\overline{\psi'}$ is ψ_1 itself so that $(\eta'_1, \dots, \eta'_r) = (\eta_1, \dots, \eta_r)$; and thus $\varphi'_1 = \varphi_1 = \varphi'$, that is $s(\varphi') = \varphi$. Therefore, $\mathbb{k}_{\tau_A^{-1} X} \subseteq \text{End}_A(\tau_A^{-1} X)$ is a \mathbb{k} -algebra section of $\kappa_{\tau_A^{-1} X}$ where $\mathbb{k}_{\tau_A^{-1} X}$ denotes the image of s .

In view of proving (a), it is useful to check that, for every $i \in \{1, \dots, r\}$, the left \mathbb{k}_{X_i} -submodule of $\text{rad}(X_i, \tau_A^{-1} X)$ generated by $g_{i,1}, \dots, g_{i,n_i}$ is a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -submodule of $\text{rad}(X_i, \tau_A^{-1} X)$. Let $\varphi \in \mathbb{k}_{\tau_A^{-1} X}$. The matter is therefore to prove that $g_{i,1}\varphi, \dots, g_{i,n_i}\varphi \in \bigoplus_{j=1}^{n_i} \mathbb{k}_{X_i} \cdot g_{i,j}$. Using the above notations in the construction of $\mathbb{k}_{\tau_A^{-1} X}$, one has $\varphi = \varphi_1$. It follows from (4) that

$$(\forall i \in \{1, \dots, r\}) \quad [g_{i,1}, \dots, g_{i,n_i}]^t \cdot \varphi = \eta_i \cdot [g_{i,1}, \dots, g_{i,n_i}]^t.$$

Since $\eta_i \in M_{n_i}(\mathbb{k}_{X_i})$, this proves the claim. The above construction therefore yields a section $\mathbb{k}_{\tau_A^{-1} X} \subseteq \text{End}_A(\tau_A^{-1} X)$ of $\kappa_{\tau_A^{-1} X}$ such that $\bigoplus_{j=1}^{n_i} \mathbb{k}_{X_i} \cdot g_{i,j}$ is a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -submodule of $\text{rad}(X_i, \tau_A^{-1} X)$, for every $i \in \{1, \dots, r\}$.

2) In order to prove (a) there only remains to define a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -linear section $\text{irr}(X_i, \tau_A^{-1} X) \rightarrow \text{rad}(X_i, \tau_A^{-1} X)$ which maps $\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}}$ to $g_{i,1}, \dots, g_{i,n_i}$, respectively, for every $i \in \{1, \dots, r\}$. Let $i \in \{1, \dots, r\}$. First, $(\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}})$ is a basis of the left \mathbb{k}_{X_i} -module $\text{irr}(X_i, \tau_A^{-1} X)$. Next, $\bigoplus_{j=1}^{n_i} \mathbb{k}_{X_i} \cdot g_{i,j}$ is a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -submodule of $\text{rad}(X_i, \tau_A^{-1} X)$. Finally, $\text{rad}^2(X_i, \tau_A^{-1} X)$ is also a $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -submodule of $\text{rad}(X_i, \tau_A^{-1} X)$. Thus the $\mathbb{k}_{X_i} - \mathbb{k}_{\tau_A^{-1} X}$ -bimodule $\text{rad}(X_i, \tau_A^{-1} X)$ is the direct sum of $(\bigoplus_{j=1}^{n_i} \mathbb{k}_{X_i} \cdot g_{i,j})$ and $\text{rad}^2(X_i, \tau_A^{-1} X)$. Whence the desired section $\text{irr}(X_i, \tau_A^{-1} X) \rightarrow \text{rad}(X_i, \tau_A^{-1} X)$. This proves (a).

3) There only remains to prove (b). According to the lemma in 1.3, the commutative diagram (2) entails that $\gamma_{\tau_A^{-1} X} = \sum_{i=1}^r \sum_{j=1}^{n_i} \overline{u f_{i,j}} \otimes \overline{g_{i,j}}$. Let $u' \in \mathbb{k}_{\tau_A^{-1} X}$ be the representative of $\overline{u} \in \kappa_{\tau_A^{-1} X}$. The image of $\gamma_{\tau_A^{-1} X}$ under the map $\bigoplus_{i=1}^r \text{irr}(X, X_i) \otimes_{\kappa_{X_i}}$

$\text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{Hom}_A(X, \tau_A^{-1}X)$ induced by all the considered sections is therefore $\sum_{i,j} u' f_{i,j} g_{i,j} = u' \sum_{i,j} f_{i,j} g_{i,j} = u' fg = 0$. \square

2.4. Inductive construction of well-behaved functors. In view of proving the existence of well-behaved functors $\mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ it is necessary to consider \mathbb{k} -linear functors $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ where χ is a full and convex subquiver of $\tilde{\Gamma}$. Here the notation $\mathbb{k}\chi$ stands for the full subcategory of $\mathbb{k}\tilde{\Gamma}$ with object set the set of vertices in χ . Following 2.2, such a functor $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ is called well-behaved if: (a) $Fx = \pi x$ for every vertex $x \in \chi$; (b) the \mathbb{k} -algebra homomorphism $\kappa_x \hookrightarrow \mathbb{k}\chi(x, x) \rightarrow \text{End}_A(\pi x)$ is a section of the natural surjection $\text{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x} = \kappa_x$, for every vertex $x \in \chi$ (as above, the image of the section is denoted by \mathbb{k}_x); (c) the \mathbb{k} -linear composite map $\text{irr}(\pi x, \pi y) = M(x, y) \hookrightarrow \mathbb{k}\chi(x, y) \rightarrow \text{Hom}_A(\pi x, \pi y)$ is a $\mathbb{k}_x - \mathbb{k}_y$ -linear section, for every arrow $x \rightarrow y$ in χ ; and (d) it vanishes on $\gamma_x = \gamma_{\pi x}$ for every non-projective vertex $x \in \chi$ such that $\tau x \in \chi$.

Lemma. *Assume that $\tilde{\Gamma}$ is with length. Let ℓ be a length function on the vertices in $\tilde{\Gamma}$. Let $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ be a well-behaved functor where $\chi \subseteq \tilde{\Gamma}$ is a full and convex subquiver distinct from $\tilde{\Gamma}$ and satisfying the following two conditions*

- (a) *either there is no arrow $x \rightarrow x_1$ in $\tilde{\Gamma}$ such that $x_1 \in \chi$ and $x \notin \chi$, or else there is an upper bound on the integers $\ell(x)$ where x runs through the vertices in $\tilde{\Gamma} \setminus \chi$ such that there exists an arrow $x \rightarrow x_1$ in $\tilde{\Gamma}$ satisfying $x_1 \in \chi$,*
- (b) *either there is no arrow $x_1 \rightarrow x$ in $\tilde{\Gamma}$ such that $x_1 \in \chi$ and $x \notin \chi$, or else there is a lower bound on the integers $\ell(x)$ where x runs through the vertices in $\tilde{\Gamma} \setminus \chi$ such that there exists an arrow $x_1 \rightarrow x$ in $\tilde{\Gamma}$ satisfying $x_1 \in \chi$.*

Then there exist a full and convex subquiver $\chi' \subseteq \tilde{\Gamma}$ which contains χ strictly and satisfies (a) and (b), and a well-behaved functor $F': \mathbb{k}\chi' \rightarrow \text{ind } \Gamma$ which extends F .

Proof. There exists an arrow $x \rightarrow x_1$ or $x_1 \rightarrow x$ in $\tilde{\Gamma}$ such that $x \notin \chi$ and $x_1 \in \chi$ because $\chi \subsetneq \tilde{\Gamma}$. Assume the former (the latter is dealt with similarly) and choose x so that $\ell(x)$ is maximal ((a)). Let χ' be the full subquiver of $\tilde{\Gamma}$ with vertices x and those of χ . Then χ' is convex in $\tilde{\Gamma}$ because $\ell(x)$ is maximal. Let $x \rightarrow x_1, \dots, x \rightarrow x_r$ be the arrows in $\tilde{\Gamma}$ starting in x and ending in some vertex in χ . Note that if x is non-injective and $\tau^{-1}x \in \chi$ then these are all the arrows in $\tilde{\Gamma}$ starting in x because $\ell(x)$ is maximal. In order to extend $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ to a functor $F': \mathbb{k}\chi' \rightarrow \text{ind } \Gamma$ distinguish two cases according to whether x is non-injective and $\tau^{-1}x \in \chi$, or not. For every $i \in \{1, \dots, r\}$ let $\mathbb{k}_i \subseteq \text{End}_A(\pi x_i)$ be the section $F(\kappa_{x_i})$ of $\kappa_{\pi x_i}$.

Assume first that either x is injective, or else x is non-injective and $\tau^{-1}x \notin \chi$. In particular, if a non-projective vertex $y \in \tilde{\Gamma}$ is such that both y and τy lie in χ' then they both lie in χ . Fix any \mathbb{k} -algebra section $\mathbb{k}_x \subseteq \text{End}_A(\pi x)$ of $\kappa_{\pi x}$. Let $i \in \{1, \dots, r\}$. The \mathbb{k} -algebra isomorphisms $\kappa_{x_i} \rightarrow \mathbb{k}_i$ (induced by F) and $\kappa_x \rightarrow \mathbb{k}_x$ allow one to consider $M(x, x_i)$ as a $\mathbb{k}_x - \mathbb{k}_i$ -bimodule. For this structure, the quotient map $\text{Hom}_A(\pi x, \pi x_i) = \text{rad}(\pi x, \pi x_i) \xrightarrow{p_i} \text{irr}(\pi x, \pi x_i)$ is $\mathbb{k}_x - \mathbb{k}_i$ -linear. On the other hand the \mathbb{k} -algebra $\mathbb{k}_x \otimes_k \mathbb{k}_i^{op}$ is semi-simple. Hence there exists

a $\mathbb{k}_x - \mathbb{k}_i$ -linear section $M(x, x_i) = \text{irr}(\pi x, \pi x_i) \xrightarrow{s_i} \text{Hom}_A(\pi x, \pi x_i)$ of p_i . The sections $\mathbb{k}_x \subseteq \text{End}_A(\pi x)$ and s_i ($i \in \{1, \dots, r\}$) extend $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ to a \mathbb{k} -linear functor $F': \mathbb{k}\chi' \rightarrow \text{ind } \Gamma$ satisfying the conditions (a), (b) and (c) in the definition of well-behaved functors. Moreover, condition (d) is satisfied for F' because it is so

for F and because, if $y \in \tilde{\Gamma}$ is non-projective and $y, \tau y \in \chi'$ then $y, \tau y \in \chi$. This proves the lemma when either x is injective or else x is non-injective and $\tau^{-1}x \notin \chi$.

Assume now that x is non-injective and $\tau^{-1}x \in \chi$. The mesh in $\tilde{\Gamma}$ starting in x has the form

$$\begin{array}{ccccc} & & x_1 & & \\ & \nearrow & \vdots & \searrow & \\ x & & & & \tau^{-1}x. \\ & \searrow & x_r & \nearrow & \end{array}$$

For simplicity let $X = \pi x$ and $X_i = \pi x_i$ for every $i \in \{1, \dots, r\}$ so that $\pi \tau^{-1}x = \tau_A^{-1}X$. Let $n_1, \dots, n_r \geq 1$ be the integers such that there is an almost split sequence $0 \rightarrow X \rightarrow \bigoplus_{i=1}^r X_i^{n_i} \rightarrow \tau_A^{-1}X \rightarrow 0$. Since $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ is well-behaved, it induces a \mathbb{k} -algebra section $\mathbb{k}_x \subseteq \text{End}_A(X)$ of $\kappa_x = \kappa_X$. It also induces a $\mathbb{k}_x - \mathbb{k}_i$ -linear section $M(x, x_i) = \text{irr}(X, X_i) \rightarrow \text{rad}(X, X_i)$, for every $i \in \{1, \dots, r\}$. Let $i \in \{1, \dots, r\}$, let $\{f_{i,j}: X \rightarrow X_i\}_{1 \leq j \leq n_i}$ be the image under this $\mathbb{k}_X - \mathbb{k}_{X_i}$ -linear section of a basis of the \mathbb{k}_{X_i} -vector space $M(x, x_i)$. Thus, the morphism $X \xrightarrow{f := [f_{i,j}; i, j]} \bigoplus_{i=1}^r X_i^{n_i}$ is left minimal almost split. Let $g = [g_{i,j}; i, j]^t: \bigoplus_{i=1}^r X_i^{n_i} \rightarrow \tau_A^{-1}X$ be its cokernel. Then 2.3 yields a \mathbb{k} -algebra section $\mathbb{k}_{\tau^{-1}x} \subseteq \text{End}_A(\tau_A^{-1}X)$ of $\kappa_{\tau^{-1}x}$ together with $\mathbb{k}_i - \mathbb{k}_{\tau^{-1}x}$ -linear sections $M(x_i, \tau^{-1}x) = \text{irr}(X_i, \tau_A^{-1}X) \rightarrow \text{rad}(X_i, \tau_A^{-1}X)$, for every $i \in \{1, \dots, r\}$, such that the induced map

$$\bigoplus_{i=1}^r M(x, x_i) \otimes_{\kappa_{x_i}} M(x_i, \tau^{-1}x) \longrightarrow \text{Hom}_A(X, \tau_A^{-1}X)$$

vanishes on $\gamma_{\tau_A^{-1}X}$. These new sections clearly extend $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ to a well-behaved functor $F': \mathbb{k}\chi' \rightarrow \text{ind } \Gamma$. Like in the previous case $F': \mathbb{k}\chi' \rightarrow \text{ind } \Gamma$ is well-behaved. Finally χ' satisfies both conditions (a) and (b) in the statement of the lemma for $\chi' \setminus \chi$ consists of one vertex. \square

2.5. Existence of well-behaved functors. The following implies Theorem A.

Proposition. *Let A be a finite-dimensional algebra over a perfect field \mathbb{k} . Let Γ be an Auslander-Reiten component of A . Let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers where $\tilde{\Gamma}$ is with length. Endow Γ and $\tilde{\Gamma}$ with \mathbb{k} -modulations as in 1.3 and 1.5, respectively.*

- (1) *There exists a well-behaved functor $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$.*
- (2) *Let $X \in \Gamma$ and let $f: X \rightarrow \bigoplus_{i=1}^r X_i^{n_i}$ be a strongly irreducible morphism. Let $x \in \pi^{-1}(X)$ and let*

$$\begin{array}{ccccc} & & x_1 & & \\ & \nearrow & \vdots & \searrow & \\ x & & & & \\ & \searrow & x_r & \nearrow & \end{array}$$

be the full subquiver of $\tilde{\Gamma}$ such that $\pi x_i = X_i$. Then there exists a well-behaved functor $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ such that F maps $\overline{f_{i,j}} \in \mathbb{k}(\tilde{\Gamma})(x, x_i)$ to $f_{i,j}$ for every $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\}$.

Proof. There always exists an irreducible morphism between indecomposable modules lying in Γ and it is strongly irreducible. Thus only (2) needs a proof. Let Σ be the set of pairs $(\chi, F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma)$ where $\chi \subseteq \tilde{\Gamma}$ is a full and convex subquiver containing x_1, \dots, x_r, x which satisfies conditions (a) and (b) in 2.4, and

$F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$ is a well-behaved functor mapping $\overline{f_{i,j}} \in \mathbb{k}(\tilde{\Gamma})(x, x_i)$ to $f_{i,j}$ for all i, j . Then Σ is ordered: $(\chi, F) \leq (\chi', F')$ if and only if $\chi \subseteq \chi'$ and F' restricts to F . Also, Σ is not empty: Let $\chi \subseteq \tilde{\Gamma}$ be the full subquiver drawn in the statement of the lemma. Fix \mathbb{k} -algebra sections $\mathbb{k}_X \subseteq \text{End}_A(X)$, and $\mathbb{k}_{X_i} \subseteq \text{End}_A(X_i)$ of κ_X and κ_{X_i} , respectively, for every $i \in \{1, \dots, n_i\}$. For every i the family $\{f_{i,j}\}_{j \in \{1, \dots, n_i\}}$ in $\text{irr}(X, X_i)$ is free over $\mathbb{k}_X \otimes_{\mathbb{k}} \mathbb{k}_{X_i}^{op}$. Hence there exists a $\mathbb{k}_X - \mathbb{k}_{X_i}$ -linear section $\text{irr}(X, X_i) \rightarrow \text{rad}(X, X_i)$ which maps $\overline{f_{i,j}}$ to $f_{i,j}$ for every $j \in \{1, \dots, n_i\}$ because $\mathbb{k}_X \otimes_{\mathbb{k}} \mathbb{k}_{X_i}^{op}$ is a semi-simple \mathbb{k} -algebra. All these sections define a well-behaved functor $F: \mathbb{k}\chi \rightarrow \text{ind } \Gamma$. Then $(\chi, F) \in \Sigma$. Note that the conditions (a) and (b) in 2.4 are satisfied because χ has only finitely many vertices. Let (χ, F) be a maximal element of Σ . Then $\chi = \tilde{\Gamma}$ (2.4). The functor $F: \mathbb{k}\tilde{\Gamma} \rightarrow \text{ind } \Gamma$ then induces a well-behaved functor $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ which fits the conclusion of the proposition. \square

2.6. Covering property of well-behaved functors. Theorem B is an adaptation of [CLMT11, Thm. B] to perfect fields.

Proof of Theorem B. The proof uses a specific left minimal almost split morphism and a specific almost split sequence that arise from F and which are now introduced. Let $X = \pi x$. For every arrow in $\tilde{\Gamma}$ with source x (say, with target x'), fix one basis over $\kappa_{\pi x'}$ of $\text{irr}(\pi x, \pi x')$. Putting these bases together (for all the arrows in $\tilde{\Gamma}$ with source x) yields a sequence of morphisms in $\mathbb{k}(\tilde{\Gamma})$ with domain x . Say, the sequence is $(\alpha_i)_{i=1, \dots, r}$ where the codomain of α_i is denoted by x_i (there may be repetitions in the sequence of codomains). Set $X_i = \pi x_i$ and $a_i = F(\alpha_i)$ for every i . In particular $a_i: X \rightarrow X_i$ is an irreducible morphism and $\overline{a_i} = \alpha_i$ if $\overline{a_i}$ is considered as lying in $\mathbb{k}(\tilde{\Gamma})(x, x_i)$. By construction, $[a_1, \dots, a_r]: X \rightarrow \bigoplus_{i=1}^r X_i$ is a left minimal almost split morphism. If x is non-injective then a completes into an almost split sequence $0 \rightarrow X \xrightarrow{a} \bigoplus_{i=1}^r X_i \xrightarrow{b} \tau_A^{-1} X \rightarrow 0$ as follows. For every $X' \in \{X_1, \dots, X_r\}$ the family $\{\overline{a_i}\}_{i \text{ s.t. } X'=X_i}$ is a basis of $\text{irr}(X, X')$ over $\kappa_{X'}$; let $\{\beta_i\}_{i \text{ s.t. } X'=X_i}$ be the corresponding dual basis of $\text{irr}(X', \tau_A^{-1} X)$ over $\kappa_{X'}$ (for the \mathbb{k} -modulation on Γ); For every $i \in \{1, \dots, r\}$ such that $X' = X_i$ set $b_i: X_i \rightarrow \tau_A^{-1} X$ to be the image of $\beta_i \in \mathbb{k}(\tilde{\Gamma})(x_i, \tau^{-1} x)$ under F (hence, if one considers $\overline{b_i}$ as lying in $\mathbb{k}(\tilde{\Gamma})(x_i, \tau^{-1} x)$ then $\overline{b_i} = \beta_i$). By construction $\gamma_{\tau^{-1} x} = \sum_{i=1}^r \overline{a_i} \otimes \overline{b_i}$. Therefore $\sum_{i=1}^r a_i b_i = 0$ because F is well-behaved and maps each $\overline{a_i}$ and $\overline{b_i}$ to a_i and b_i , respectively. Since moreover a is left minimal almost split, the morphism $b = [b_1, \dots, b_r]^t$ is right minimal almost split. Thus (a, b) forms the announced almost split sequence.

(a) The two maps are dual to each other so only the first one is taken care of.

The surjectivity for every x is proved by induction on $n \geq 0$. If $n = 0$ it follows from: $\text{rad}^0(Fx, Fy)/\text{rad}(Fx, Fy)$ is κ_{Fx} or 0 according to whether $Fx = Fy$ or $Fx \neq Fy$, and $\mathcal{R}^0 \mathbb{k}(\tilde{\Gamma})(x, z)/\mathcal{R} \mathbb{k}(\tilde{\Gamma})(x, z)$ is κ_x or 0 according to whether $x = z$ or $x \neq z$. If $n \geq 1$, if the surjectivity is already proved for indices smaller than n , and if $f \in \text{rad}^n(Fx, Fy)$ is given, then there exists $(u_i)_i \in \bigoplus_{i=1}^r \text{rad}^{n-1}(Fx_i, Fy)$ such that $f = \sum_i a_i u_i$ (1.2); for every i , there exists $(\beta_{i,z})_z \in \bigoplus_{Fz=Fy} \mathcal{R}^{n-1} \mathbb{k}(\tilde{\Gamma})(x_i, z)$ such that $u_i = \sum_z F(\beta_{i,z}) \text{ mod } \text{rad}^n$ (induction hypothesis); thus $f - \sum_z F(\sum_i \alpha_i \beta_{i,z}) \in \text{rad}^{n+1}$. This proves the surjectivity at index n .

The injectivity for every x is also proved by induction on $n \geq 0$. If $n = 0$ it follows from: $\mathbb{k}(\tilde{\Gamma})(x, z) = \mathcal{R} \mathbb{k}(\tilde{\Gamma})(x, z)$ if $x \neq z$, and $\mathbb{k}(\tilde{\Gamma})(x, x) = \kappa_x = \kappa_{Fx}$, and F induces

a section $\kappa_x \rightarrow \text{End}_A(Fx)$ of the canonical surjection $\text{End}_A(Fx) \rightarrow \kappa_{Fx}$. Let $n \geq 1$. Assume the injectivity for indices smaller than n . Let $(\phi_z) \in \oplus_{Fz=Fly} \mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})(x, z)$ be such that $\sum_z F(\phi_z) \in \text{rad}^{n+1}(Fx, Fly)$. Using the surjectivity and 1.2 yields $(\psi_z)_z \in \oplus_{Fz=Fly} \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})(x, z)$ together with $(u_i)_i \in \oplus_{i=1}^r \text{rad}^{n+1}(Fx_i, Fly)$ such that $\sum_z F(\phi_z - \psi_z) = \sum_i a_i u_i$. On the other hand, $n \geq 1$ and $\{\alpha_j\}_{j \in \{1, \dots, r\}}$ contains a basis of $M(x, x_i)$ over κ_{x_i} for every $i \in \{1, \dots, r\}$. The construction of $\mathbb{k}(\tilde{\Gamma})$ therefore yields $(\theta_{i,z})_i \in \oplus_{i=1}^r \mathbb{k}(\tilde{\Gamma})(x_i, z)$ such that $\phi_z - \psi_z = \sum_i \alpha_i \theta_{i,z}$, for every z . Putting these morphisms together and using that $a_i = F(\alpha_i)$ for every i yields $\sum_i a_i (\sum_z F(\theta_{i,z}) - u_i) = 0$. Now distinguish two cases according to whether x is injective or not. If x is injective then $\sum_z F(\theta_{i,z}) = u_i$ which, following the induction hypothesis, implies that $\theta_{i,z} \in \mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})$ for every i and every z . Thus $\phi_z = \psi_z + \sum_i \alpha_i \theta_{i,z} \in \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})$ for every z . If x is not injective there exists $v \in \text{Hom}_A(\tau_A^{-1} Fx, Fly)$ such that $\sum_z F(\theta_{i,z}) - u_i = b_i v$ for every i . Using again the surjectivity yields $(\chi_z)_z \in \oplus_{Fz=Fly} \mathbb{k}(\tilde{\Gamma})(\tau^{-1}x, z)$ such that $v = \sum_z F(\chi_z) \text{ mod } \text{rad}^{n-1}$. In particular $b_i v = \sum_z F(\beta_i \chi_z) \text{ mod } \text{rad}^n$ for every i . Hence $\sum_z F(\theta_{i,z} - \beta_i \chi_z) = u_i \text{ mod } \text{rad}^n$. Therefore $\theta_{i,z} - \beta_i \chi_z \in \mathfrak{R}^n \mathbb{k}(\tilde{\Gamma})$ for every i and every z (by induction and because $u_i \in \text{rad}^{n+1}$). Since moreover $\sum_i \alpha_i \beta_i = 0$, $\psi_z \in \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})$ and $\phi_z = \psi_z + \sum_i \alpha_i \theta_{i,z}$ for every z , it follows that $\phi_z \in \mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})$.

(b) follows from (a) and from 1.4 (part (b)).

(c) follows from (a), (b) and the fact that Γ is generalised standard, that is, $\bigcap_{n \geq 0} \text{rad}^n(X, Y) = 0$ for every $X, Y \in \Gamma$. \square

3. APPLICATION TO COMPOSITIONS OF IRREDUCIBLE MORPHISMS

The following equivalence was proved in [CLMT11, Prop. 5.1] when \mathbb{k} is algebraically closed and under the additional assumption that the valuation of the involved arrows are trivial. This last assumption is dropped here.

Proposition. *Let $X_1, \dots, X_{n+1} \in \text{ind } A$. The following conditions are equivalent*

- (a) *there exist irreducible morphisms $X_1 \xrightarrow{h_1} \dots \xrightarrow{h_n} X_n$ such that $h_1 \cdots h_n \in \text{rad}^{n+1} \setminus \{0\}$,*
- (b) *there exist irreducible morphisms $f_i: X_i \rightarrow X_{i+1}$ and morphisms $\varepsilon_i: X_i \rightarrow X_{i+1}$, for every i , such that $f_1 \cdots f_n = 0$, such that $\varepsilon_1 \cdots \varepsilon_n \neq 0$ and such that, for every i , either $\varepsilon_i \in \text{rad}^2$ or else $\varepsilon_i = f_i$.*

Proof. The implication (b) \Rightarrow (a) was proved in [CT10, Thm. 2.7] (the proof there works for artin algebras and the standard hypothesis made there plays no role for this implication). Assume (a). Let Γ be the component of $\Gamma(\text{mod } A)$ containing X_1, \dots, X_{n+1} , let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be the universal covering and $F: \mathbb{k}(\tilde{\Gamma}) \rightarrow \text{ind } \Gamma$ be a well-behaved functor (2.5). Let $x_1 \in \pi^{-1}(X_1)$. There is a unique path $\gamma: x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{n+1}$ in $\tilde{\Gamma}$ which image under π is $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{n+1}$. Let $h_i: X_i \rightarrow X_{i+1}$ ($1 \leq i \leq n$) be irreducible morphisms such that $h_1 \cdots h_n \in \text{rad}^{n+1} \setminus \{0\}$ and consider $\overline{h_i} \in \text{irr}(X_i, X_{i+1})$ as lying in $\mathbb{k}(\tilde{\Gamma})(x_i, x_{i+1})$. Let $h'_i = h_i - F(\overline{h_i})$ for $1 \leq i \leq n$. Then $h'_i \in \text{rad}^2$ because F is well-behaved. Therefore $F(\overline{h_1} \cdots \overline{h_n}) \in \text{rad}^{n+1}$. Since $\mathfrak{R}^{n+1} \mathbb{k}(\tilde{\Gamma})(x_1, x_{n+1}) = 0$ (the path γ has length n , 1.4), it follows that $\overline{h_1} \cdots \overline{h_n} = 0$ (2.6). This and $h_1 \cdots h_n \neq 0$ imply that

$F(\overline{h_1} \cdots \overline{h_n}) - h_1 \cdots h_n \neq 0$ that is, the sum of the morphisms

$$F(\overline{h_1}) \cdots F(\overline{h_{i_1-1}}) h'_{i_1} F(\overline{h_{i_1+1}}) \cdots F(\overline{h_{i_t-1}}) h'_{i_t} F(\overline{h_{i_t+1}}) \cdots F(\overline{h_n}),$$

for $t \in \{1, \dots, n\}$ and $1 \leq i_1 < \cdots < i_t \leq n$, is non-zero. Hence there exists $t \in \{1, \dots, n\}$ and $1 \leq i_1 < \cdots < i_t \leq n$ such that the corresponding term in the above sum is non-zero. Define $f_j := F(\overline{h_j})$, and $\varepsilon_j := F(\overline{h_j})$ if $j \notin \{i_1, \dots, i_t\}$ or $\varepsilon_j := h'_j$ if $j \in \{i_1, \dots, i_t\}$. Then $\{f_i, \varepsilon_i\}_{i=1, \dots, n}$ fits the requirements of (b). \square

4. ACKNOWLEDGEMENTS

Part of the work presented in this text was done while the second named author was visiting Université de Sherbrooke (Québec, Canada). He thanks Ibrahim Assem and the Department of Mathematics there for the warm hospitality.

REFERENCES

- [ARO97] Maurice Auslander, Idun Reiten, and Smalø Sverre O. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.
- [BG82] Klaus Bongartz and Pierre Gabriel. Covering spaces in representation-theory. *Invent. Math.*, 65(3):331–378, 1981/82.
- [Cha10] Claudia Chaio. Degrees of irreducible morphisms in standard components. *J. Pure Appl. Algebra*, 214(7):1063–1075, 2010.
- [CLMT11] Claudia Chaio, Patrick Le Meur, and Sonia Trepode. Degrees of irreducible morphisms and finite-representation type. *J. Lond. Math. Soc. (2)*, 84(1):35–57, 2011.
- [CT10] Claudia Chaio and Sonia Trepode. The composite of irreducible morphisms in standard components. *J. Algebra*, 323(4):1000–1011, 2010.
- [IT84a] Kiyoshi Igusa and Gordana Todorov. A characterization of finite Auslander-Reiten quivers. *J. Algebra*, 89(1):148–177, 1984.
- [IT84b] Kiyoshi Igusa and Gordana Todorov. Radical layers of representable functors. *J. Algebra*, 89(1):105–147, 1984.
- [Liu92] Shiping Liu. Degrees of irreducible maps and the shapes of Auslander-Reiten quivers. *J. London Math. Soc. (2)*, 45(1):32–54, 1992.
- [Rie80] Christine Riedtmann. Algebren, Darstellungsköcher, Überlagerungen und zurück. *Comment. Math. Helv.*, 55(2):199–224, 1980.

(Claudia Chaio) DEPARTAMENTO DE MATEMÁTICA, FCEyN, UNIVERSIDAD NACIONAL DE MAR DEL PLATA, FUNES 3350, 7600 MAR DEL PLATA, ARGENTINA
E-mail address: `claudia.chaio@gmail.com`

(Patrick Le Meur) LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ BLAISE PASCAL & CNRS, COMPLEXE SCIENTIFIQUE LES CÉZEAUX, BP 80026, 63171 AUBIÈRE CEDEX, FRANCE
Current address: Université Paris Diderot, Sorbonne Paris Cité, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Univ. Paris 06, F-75013, Paris, France
E-mail address: `patrick.le-meur@imj-prg.fr`

(Sonia Trepode) DEPARTAMENTO DE MATEMÁTICA, FCEyN, UNIVERSIDAD NACIONAL DE MAR DEL PLATA, FUNES 3350, 7600 MAR DEL PLATA, ARGENTINA
E-mail address: `strepode@mdp.edu.ar`